Math 5C Test 3 Spring 2024

Follow Instructions given on Canvas.

(1) Evaluate $\int_1^4 \int_0^{\sqrt{x}} x\,dy\,dx$

(10 points)

$$= \int_1^4 x\,y\Big]_0^{\sqrt{x}} dx = \int_1^4 x\sqrt{x}\,dx$$

$$= \int_1^4 x^{3/2}dx = \frac{2}{5}x^{5/2} = \frac{2}{5}\left(4^{5/2}-1\right)$$

$$= \frac{2}{5}(32-1) = \frac{62}{5}$$

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(2) The integral $\iint_R (8-2x-2y)\,dA$ where R=[0,1]X[0,2] can be used to find the volume of a solid.

Sketch the solid, showing scale. (5 points)

$$\boxed{f(x,y) = 8-2x-2y}$$

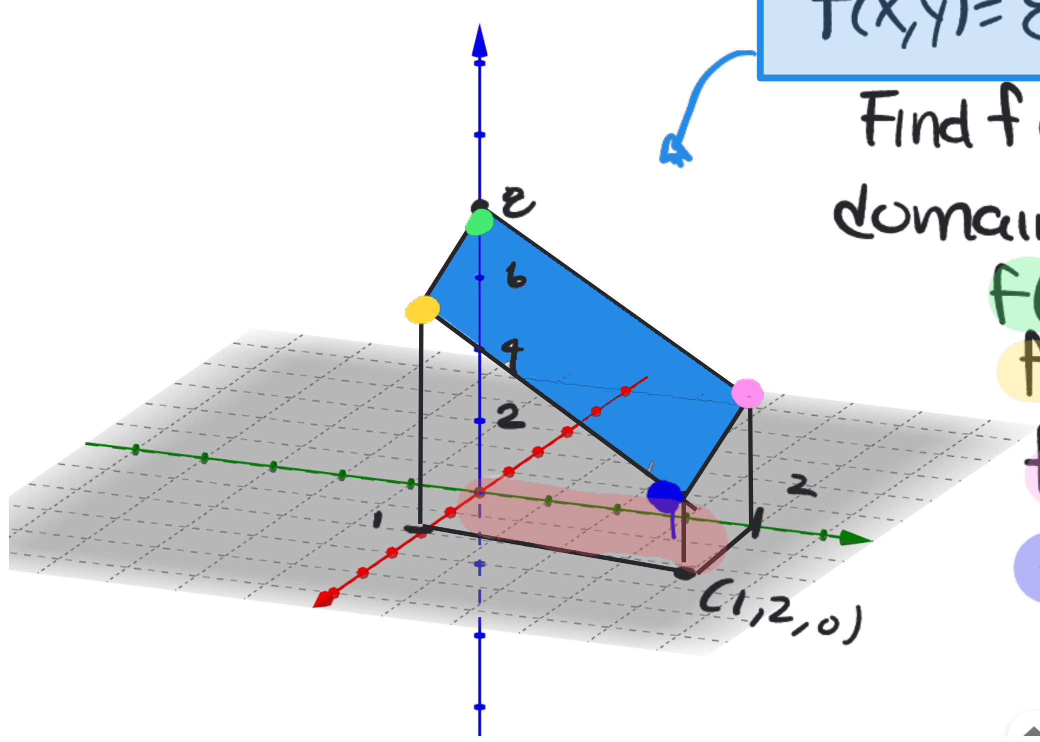
Find f at corners of domain

$f(0,0) = 8$

$f(1,0) = 6$

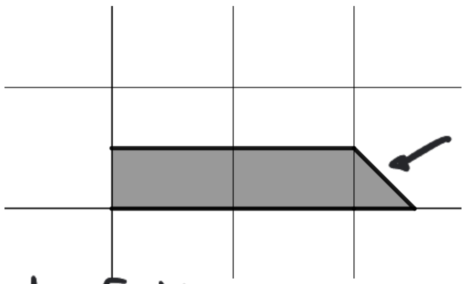
$f(0,2) = 4$

$f(1,2) = 2$

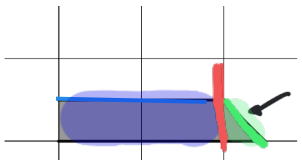


See 15.1 video #2, and 15.1 HW problems 35, 37

(3) Evaluate $\iint_D y^3\,dA$ where D is the region shown, bounded by $y=1$ and $y=5-x$, and the coordinate axes. (10 points)



$y=5-x$
$x=5-y$



If you choose to do dy first, you would have to split into 2 integrals

$$\int_0^1\int_0^{5-y} y^3\,dx\,dy = \int_0^1 (5-y)y^3\,dy$$

$$= \int_0^1 (5y^3-y^4)\,dy = \frac{5}{4}y^4-\frac{1}{5}y^5\Big]_0^1 = \frac{5}{4}-\frac{1}{5} = \frac{21}{20}$$

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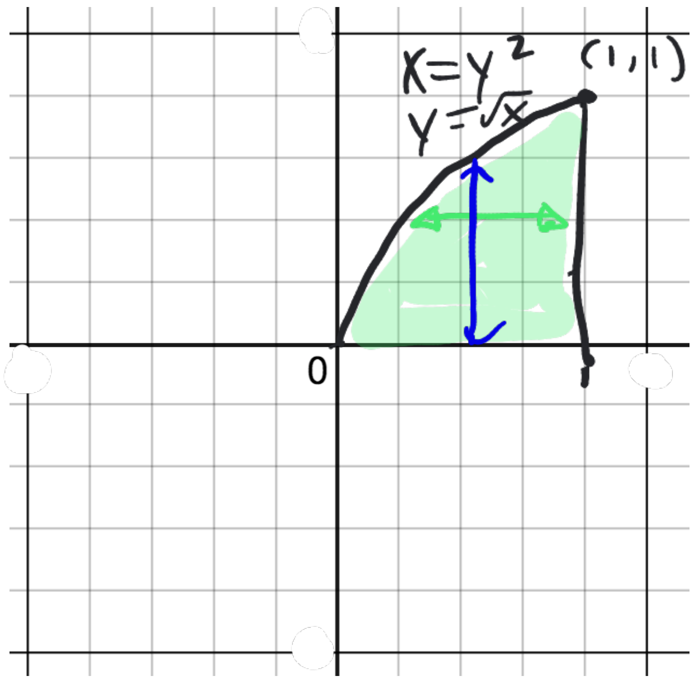
(4) Compute $\int_0^1\int_{y^2}^1 \sqrt{x}\sin x\,dx\,dy$

$y^2 \le x \le 1$

left right

You may want to reverse the order of integration. *(10 points)*



$$\int_0^1\int_{y^2}^1 \sqrt{x}\sin x\,dx\,dy$$

$$= \int_0^1\int_0^{\sqrt{x}} \sqrt{x}\sin x\,dy\,dx$$

$$= \int_0^1 x\sin x\,dx$$

by parts $u=x$ $dv=\sin x\,dx$
$du=dx$ $v=-\cos x$

$$= -x\cos x+\sin x\Big]_0^1$$

$$= -\cos(1)+\sin(1)$$

(5) $\int_C yz\cos x\, ds$ where C is given by x=t, y=2cost, z=2sint, *(10 points)*

$0 \le t \le \pi$

$\vec{r}(t) = \langle t, 2\cos t, 2\sin t \rangle$

$\vec{r}\,'(t) = \langle 1, -2\sin t, 2\cos t \rangle$

$ds = \|\vec{r}\,'(t)\| dt = \sqrt{1+4\sin^2 t + 4\cos^2 t}\, dt$

$\sqrt{1+4}\, dt$

$\sqrt{5}\, dt$

SO $\int_C yz\cos x\, ds = \sqrt{5}\int_0^{\pi} 2\cos t\, 2\sin t \cos t\, dt$

$4\sqrt{5}\int_0^{\pi} \cos^2 t \sin t\, dt$ $\quad u = \cos t$, $du = \sin t\, d(-$

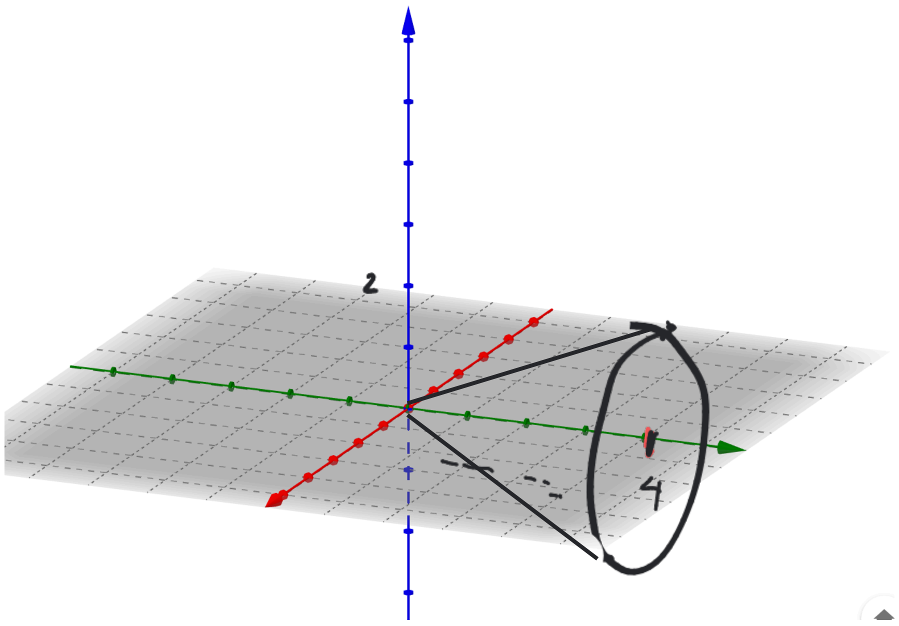
$-4\sqrt{5}\int_1^{-1} u^2 du$

$-\frac{4\sqrt{5}}{3}\, u^3 \Big]_1^{-1}$

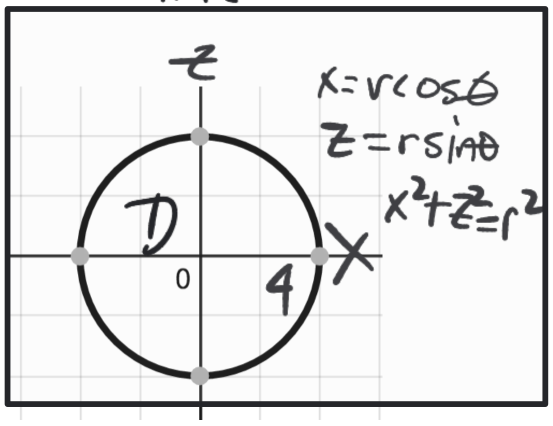
$\frac{8\sqrt{5}}{3}$

(6) Evaluate $\iint_S y\,dS$ where S is the ~~portion~~ of the paraboloid $\cancel{y = x^2+z^2}$, $0 \le y \le 4$

Cone $\quad y = \sqrt{x^2+z^2}$ *(10 points)*



$$g_x = \frac{x}{\sqrt{x^2+z^2}} \qquad g_z = \frac{z}{\sqrt{x^2+z^2}}$$



$$dS = \sqrt{g_x^2 + g_z^2 + 1}\;dA = \sqrt{\frac{x^2}{x^2+z^2} + \frac{z^2}{x^2+z^2} + 1}\;dA$$

$$= \sqrt{\frac{x^2+z^2}{x^2+z^2} + 1}\;dA = \sqrt{2}\,dA$$

$$\iint_S y\,dS = \iint_D \sqrt{x^2+z^2}\;\sqrt{2}\,dA$$

$$= \int_0^{2\pi}\int_0^4 r\;\sqrt{2}\;r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^4 r^2\,dr\,d\theta = \frac{\sqrt{2}}{3}\int_0^{2\pi} r^3\Big]_0^4\,d\theta$$

$$= \frac{\sqrt{2}}{3}\cdot 64\int_0^{2\pi} d\theta = \frac{\sqrt{2}}{3}\cdot 64\cdot 2\pi = \frac{128\sqrt{2}\,\pi}{3}$$

(7) SET UP BUT DO NOT EVALUATE: integrals as specified to find the volume enclosed by the cone $z = \frac{1}{2}\sqrt{x^2+y^2}$ and the plane z = 5 *in the first octant.*

grids avail for use

(20 points)

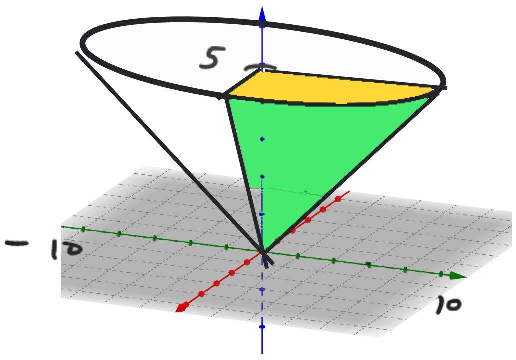
a) Sketch the solid neat

intersection

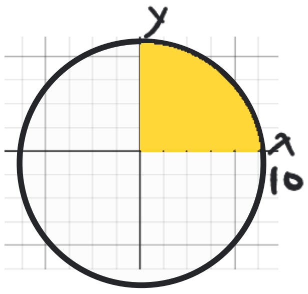
$5 = \frac{1}{2}\sqrt{x^2+y^2}$

$10 = \sqrt{x^2+y^2}$

$x^2+y^2=100$



b) Triple integral - cylindrical coordinates.



$$\int_0^{\pi/2}\int_0^{10}\int_{\frac{1}{2}r}^{5} dz\, r\,dr\,d\theta$$

Convert z=5 to cylindrical

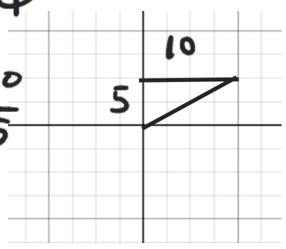
$\rho\cos\phi = 5$

$\rho = 5/\cos\phi$

c) Triple integral - spherical coordinates.

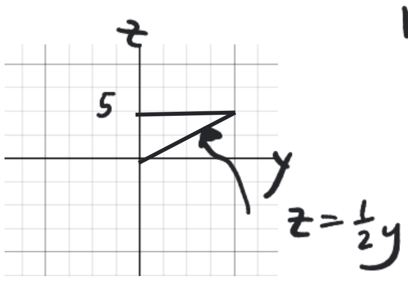
Find ϕ

$\tan\phi = \frac{10}{5}$



$$\int_0^{\pi/2}\int_0^{\tan^{-1}2}\int_0^{5\sec\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

d) Triple Integral – order dx dz dy



$$\int_0^{10}\int_{\frac{1}{2}y}^{5}\int_0^{\sqrt{4z^2-y^2}} dx\,dz\,dy$$

$z = \frac{1}{2}\sqrt{x^2+y^2}$

$2z = \sqrt{x^2+y^2}$

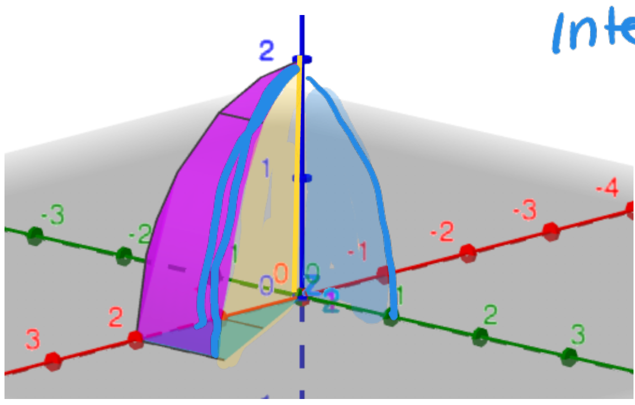
$4z^2 = x^2+y^2$

(8) SET UP BUT DO NOT EVALUATE: integrals for $\iiint_E f($[redacted]$z)\,dV$ in the order specified,

where E is the solid bound by $\begin{cases} x^2+z^2=4 \\ x=2y \\ y=0 \end{cases}$ in the first octant

In each part, sketch the necessary projection

(15 points)



Intersection

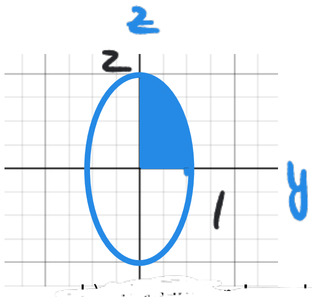
$\begin{cases} x^2+z^2=4 \\ x=2y \end{cases}$

eliminate x

$4y^2+z^2=4$

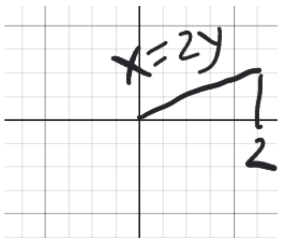
$y^2+\frac{z^2}{4}=1$

a) Triple integral – order dx dy dz



$$\int_0^2 \int_0^{\sqrt{1-\frac{z^2}{4}}} \int_{2y}^{\sqrt{4-z^2}} dx\,dy\,dz$$

Find volume



$$\int_0^2 \int_0^{\frac{1}{2}x} \int_0^{\sqrt{4-x^2}} dz\,dy\,dx$$

$$= \int_0^2 \int_0^{1/2 x} \sqrt{4-x^2}\,dy\,dx$$

$$= \int_0^2 \sqrt{4-x^2}\; y\Big]_0^{1/2 x} dx$$

$$= \int_0^2 \tfrac{1}{2}x\sqrt{4-x^2}\,dx$$

$u=4-x^2$

$du=-2x\,dx$

$$= -\frac{1}{4}\int_4^0 u^{1/2} = \frac{1}{6}u^{3/2}\Big]_0^4 = \frac{4}{3}$$

$$\int_0^{\pi/2}\int_0^2\int_0^{\frac{1}{2}r\cos\theta} dz\,r\,dr\,d\theta$$

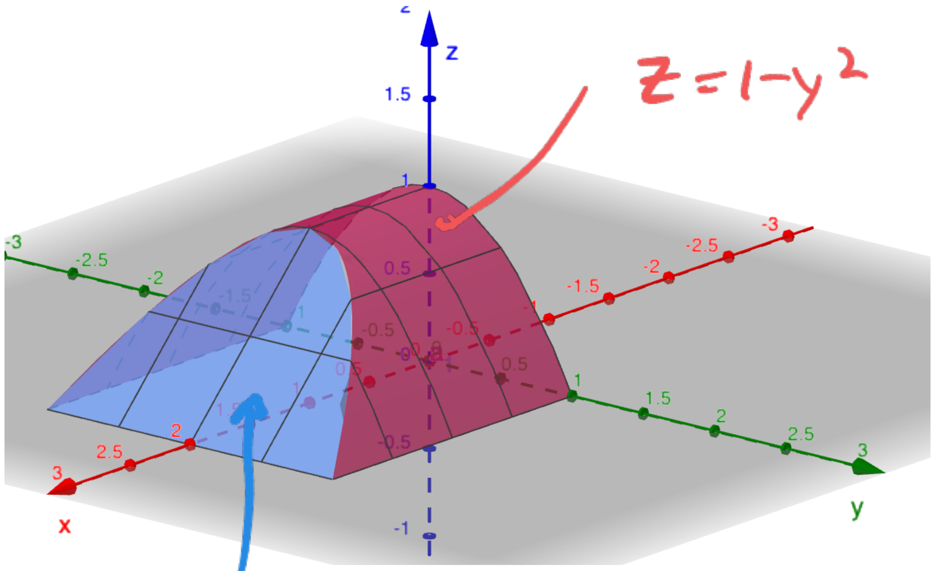
$$\int_0^{\pi/2}\int_0^2 \frac{1}{2}r^2\cos\theta\,d\theta$$

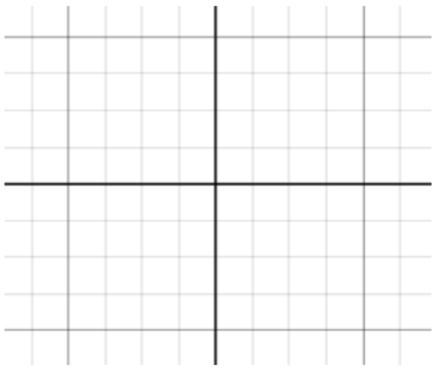
$\vdots$

$\frac{4}{3}$

(9) Find the volume of the solid bound by $z=1-y^2, x+z=2, z=0, \cancel{y}=0$, (10 points)

$z=1-y^2$





$x+z=2$

$$V=\iiint dV$$

$$=\int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{2-z} dx\,dz\,dy$$

$$=\int_{-1}^{1}\int_{0}^{1-y^2}(2-z)\,dz\,dy$$

$$\int_{-1}^{1} 2z-\tfrac{1}{2}z^2\Big]_0^{1-y^2}dy$$

$$\int_{-1}^{1}\left(2(1-y^2)-\tfrac{1}{2}(1-y^2)^2\right)dy$$

$$\int_{-1}^{1}(2-2y^2-\tfrac{1}{2}+y^2-\tfrac{1}{2}y^4\quad dy$$

(even) $2\int_0^1\left(\frac{3}{2}-y^2-\frac{1}{2}y^4\right)dy=2\left[\frac{3}{2}y-\frac{1}{3}y^3-\frac{1}{10}y^5\right]_0^1$

$$2\left(\frac{3}{2}-\frac{1}{3}-\frac{1}{10}\right)=\frac{32}{15}$$